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THE STABILITY OF EQUILIBRIUM IN CONSERVATIVE SYSTEMS*

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Chetayev's instability theorem for conservative systems is generalized to the case of a non-isolated equilibrium position. On the assumption that the potential energy is not a minimum at the equilibrium position in question, consideration is given to the Lebesgue measure of invariant sets in the intersection of the domain in which the energy integral is negative and a small neighbourhood of the equilibrium position.

Chetayev's paper /1/ played a key role in establishing a series of new cases in which the inverse of the celebrated Lagrange-Dirichlet Theorem holds (see surveys /2, 3/, Chap. III). A convincing indicator to that effect is provided, in particular, by the results obtained in /4, 5/, which utilize Chetayev's idea of constructing an auxiliary vector field possessing certain properties with regard to the potential of the forces of the system. At the same time, it has been shown /6/ that the condition according to which the function $\Pi(\mathbf{q})$ has no critical points in the domain $\omega_\varepsilon = \{\mathbf{q} \in s_\varepsilon = \{\mathbf{q} \in R^n, \|\mathbf{q}\| < \varepsilon\}: \Pi(\mathbf{q}) < 0\}$, where $\Pi(\mathbf{q})$ is the potential energy of the system, which is commonly assumed both in Chetayev's Theorem and in most further research in that area, is not essential. In many cases /6/ it may be dropped, provided certain restrictions are imposed on the structure of the set of critical points of $\Pi(\mathbf{q})$. As will be shown below, these restrictions may be lifted too.

Consider a natural system with n degrees of freedom, representing it in Hamiltonian form

$$\mathbf{q}^{\cdot} = \partial H/\partial \mathbf{p}, \quad \mathbf{p}^{\cdot} = -\partial H/\partial \mathbf{q} \tag{1}$$

$$H(\mathbf{q}, \mathbf{p}) = {}^1/_2 \mathbf{p}^T A(\mathbf{q})\,\mathbf{p} + \Pi(\mathbf{q}) = h = \text{const} \tag{2}$$

We shall assume that $H(\mathbf{q}, \mathbf{p}) \in C_{\mathbf{q}}^2(D \subset R^{2n})$, the quadratic form $\mathbf{p}^T A(0)\,\mathbf{p}$ is positive definite, the system of Eqs.(1) is in equilibrium at the point $\mathbf{q} = \mathbf{p} = 0$ and $\Pi(0) = 0$.

Theorem 1. Suppose that for as small a value of $\varepsilon > 0$ $(D \supset \bar{s}_\varepsilon)$ as desired the set $\omega_\varepsilon = \{\mathbf{q} \in s_\varepsilon: \Pi(\mathbf{q}) < 0\}$ is not empty and $0 \in \partial\omega_\varepsilon$. Assume that a vector field $\mathbf{f}(\mathbf{q}) \in C^1: s_\varepsilon \to R^n$ exists such that

1) $\mathbf{f}^T(\mathbf{q})\,\partial\Pi/\partial\mathbf{q} \leqslant 0, \quad \forall \mathbf{q} \in \omega_\varepsilon;$

2) $\mathbf{x}^T \left(\dfrac{\partial \mathbf{f}}{\partial \mathbf{q}} A(\mathbf{q})\right)\Big|_{\mathbf{q}=0} \mathbf{x} - \dfrac{1}{2}\mathbf{f}^T(0)\dfrac{\partial}{\partial \mathbf{q}}(\mathbf{x}^T A(\mathbf{q})\,x)\|_{\mathbf{q}=0} \geqslant c\|\mathbf{x}\|^2, \quad \forall \mathbf{x} \in R^n, \qquad 0 < c = \text{const}.$

Then the equilibrium position $\mathbf{q} = \mathbf{p} = 0$ of the system of Eqs.(1) is unstable.

Proof. Since by assumption $\omega_\varepsilon \neq \varnothing$, the same is true of the set

$$\Omega_\varepsilon = \{(\mathbf{q}, \mathbf{p}) \in s_\varepsilon^* = \{(\mathbf{q}, \mathbf{p}) \in R^{2n}, \|\mathbf{q} \oplus \mathbf{p}\| < \varepsilon\}: H = h < 0\}$$

Consider the auxiliary function $V = \mathbf{f}^T\mathbf{p}$. Its derivative along the vector field defined by the system of Eqs.(1) is

$$V^{\cdot} = \mathbf{p}^T \frac{\partial \mathbf{f}}{\partial \mathbf{q}} A(\mathbf{q})\,\mathbf{p} - \frac{1}{2}\mathbf{f}^T(\mathbf{q})\frac{\partial}{\partial \mathbf{q}}(\mathbf{p}^T A(\mathbf{q})\,\mathbf{p}) - \mathbf{f}^T(\mathbf{q})\frac{\partial \Pi}{\partial \mathbf{q}} \tag{3}$$

Noting that $\mathbf{f}(\mathbf{q}) \in C^1$, $A(\mathbf{q}) \in C^2$, so that the right-hand side of Eq.(3) is continuous, we express the latter in the neighbourhood of $\mathbf{q} = \mathbf{p} = 0$ in the form

$$V^{\cdot} = \mathbf{p}^T \left(\frac{\partial \mathbf{f}}{\partial \mathbf{q}} A(\mathbf{q})\right)\Big|_{\mathbf{q}=0} \mathbf{p} - \frac{1}{2}\mathbf{f}^T(0)\frac{\partial}{\partial \mathbf{q}}(\mathbf{p}^T A(\mathbf{q})\,\mathbf{p})\,|_{\mathbf{q}=0} - \mathbf{f}^T(\mathbf{q})\frac{\partial \Pi}{\partial \mathbf{q}} + o(\|\mathbf{p}\|^2) \tag{4}$$

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Using (4) and conditions 1 and 2 of the theorem, we can always choose a number η $(0<\eta<\varepsilon)$ so small that

$$V^{\cdot} \geqslant c_1 \|\mathbf{p}\|^2, \quad \forall\, (\mathbf{q}, \mathbf{p}) \in s_\eta^* \cap \Omega_\varepsilon, \quad 0<c_1<c \qquad (5)$$
$$c_1 = \text{const}$$

Suppose that $\mathbf{q}=\mathbf{p}=\mathbf{0}$ is a stable position of equilibrium. Then $\forall\, (\mathbf{q}_0, \mathbf{p}_0) \in s_\delta^* \cap \Omega_\varepsilon$, where δ is sufficiently small, there exists a positive semitrajectory $\overline{\gamma^+}(\mathbf{q}_0, \mathbf{p}_0)$ passing through a point $(\mathbf{q}_0, \mathbf{p}_0) \in s_\delta^* \cap \Omega_\varepsilon$ such that $\gamma^+(\mathbf{q}_0, \mathbf{p}_0) \subset s_\eta^* \cap \Omega_\varepsilon$. Now choose a point $(\mathbf{q}_0^*, \mathbf{p}_0^*) \in s_\delta^* \cap \Omega_\varepsilon$ so that

$$V(\mathbf{q}(t), \mathbf{p}(t))|_{t=0} = \mathbf{f}^T(\mathbf{q})\mathbf{p}|_{t=0} = \mathbf{f}^T(\mathbf{q}_0^*)\mathbf{p}_0^* = \lambda_1 > 0$$

This is equivalent to the requirement that the scalar product of the vectors $\mathbf{f}(\mathbf{q}_0^*)$ and $\mathbf{p}_0^*$ be positive in R^n. Since the fact that the point $(\mathbf{q}_0^*, \mathbf{p}_0^*)$ belongs to the domain $s_\delta^* \cap \Omega_\varepsilon$ imposes no restriction on the direction of the vector $\mathbf{p}_0^*$ in R^n (as follows at once from the structure of the energy integral $H(\mathbf{q}, \mathbf{p}) = h$), the direction may always be chosen so that the constant λ_1 is positive.

Since the solution $(\mathbf{q}^*(t), \mathbf{p}^*(t))^T$ of (1) corresponding to the semitrajectory $\gamma^+(\mathbf{q}_0^*, \mathbf{p}_0^*)$ satisfies the estimate (5), it follows that

$$V(\overline{\gamma^+}) \geqslant V(\mathbf{q}_0^*, \mathbf{p}_0^*) = \lambda_1 > 0$$

and so the point $\mathbf{p}=\mathbf{0}$ at which V vanishes is not in $\overline{\gamma^+}$. Hence it follows that $\|\mathbf{p}(\overline{\gamma^+})\|^2 \neq 0$. Now, taking into account that $\overline{\gamma^+}$ is a compact set, so that the function $\|\mathbf{p}(\overline{\gamma^+})\|^2$ attains its extremal values there, we have

$$\|\mathbf{p}(\overline{\gamma^+})\|^2 \geqslant c_2 > 0, \quad c_2 = \text{const}$$

Using this estimate we deduce from (5) that

$$V^{\cdot}(\mathbf{q}^*(t), \mathbf{p}^*(t)) \geqslant c_1 c_2 = c_3 > 0, \quad c_3 = \text{const} \qquad (6)$$

On the other hand, since $V(\mathbf{q}, \mathbf{p}) \in C^1(s_\eta^*)$ and $\bar{\gamma}^+ \subset s_\eta^*$, we have

$$V(\mathbf{q}^*(t), \mathbf{p}^*(t)) \leqslant \lambda_2, \quad 0 < \lambda_2 = \text{const} \qquad (7)$$

Comparing inequalities (6) and (7), we arrive at a contradiction, whence it follows that the equilibrium position is indeed unstable, thus proving Theorem 1.

Corollary. Suppose that for small $\varepsilon > 0$ $(D \supset \bar{s}_\varepsilon)$ $\omega_\varepsilon \neq \varnothing$, $0 \in \partial\omega_\varepsilon$ and there exists a vector field $\mathbf{f}(\mathbf{q}) \in C^1\colon s_\varepsilon \to R^n$, $\mathbf{f}(0) = 0$, such that condition 1 of Theorem 1 is satisfied and, in addition,

$$\mathbf{x}^T(\partial\mathbf{f}/\partial\mathbf{q}\, A(\mathbf{q}))|_{\mathbf{q}=0}\mathbf{x} \geqslant c\|\mathbf{x}\|^2, \quad \forall \mathbf{x} \in R^n, \quad 0 < c = \text{const}$$

Then the equilibrium position $\mathbf{q}=\mathbf{p}=\mathbf{0}$ of system (1) is unstable.

The corollary, which is a stronger version of Chetayev's Theorem /1/, since the inequality in condition 1 need not be strict, may be proved along the same lines as above. In this situation, however, it is more convenient to consider the Chetayev function $V = -H\mathbf{f}^T\mathbf{p}$, whose derivative along the vector field defined by system (1) may be expressed, using the previous arguments, in the form

$$V^{\cdot} = -H\Big(\mathbf{p}^T\Big(\frac{\partial\mathbf{f}}{\partial\mathbf{q}}A(\mathbf{q})\Big)\Big|_{\mathbf{q}=0}\mathbf{p} - \mathbf{f}^T\frac{\partial\Pi}{\partial\mathbf{q}} + o(\|\mathbf{p}\|^2)\Big) \qquad (8)$$

In the domain

$$\Lambda = \{(\mathbf{q}, \mathbf{p}) \in s_\varepsilon^*\colon H = h < 0,\ \mathbf{f}^T\mathbf{p} > 0\}$$

where $V > 0$ and $\varepsilon > 0$ is sufficiently small, an examination of the assumptions of the corollary shows that the right-hand side of (8) is positive. At $\mathbf{p}=\mathbf{0}$, where the derivative $V^{\cdot}$ may vanish, V will also vanish. Thus, if we henceforth confine our attention to trajectories of system (1) that pass through Λ and observe that

$$V = 0, \quad \forall (\mathbf{q}, \mathbf{p}) \in \partial\Lambda \cap s_\varepsilon^*; \quad V > 0, \quad V^{\cdot} > 0, \quad \forall (\mathbf{q}, \mathbf{p}) \in \Lambda$$

then the truth of the corollary follows from Chetayev's instability Theorem (/3/, p.19).

Remarks. 1. Although Chetayev, in his paper /1/, in fact considered a domain analogous to Λ, he actually postulated that $V^{\cdot}$ be positive in the domain Ω_ε defined above, which is larger than Λ and contains the latter as a proper subset. But if one requires only that $V^{\cdot}$ be positive in Λ, the instability conclusion will also hold, as might have been shown above, assuming the truth of the non-strict inequality $\mathbf{f}^T\partial\Pi/\partial\mathbf{q} \leqslant 0$ rather than the existence of the set of critical points in ω_ε without any restrictions on its structure. The function

V proposed in Chetayev's paper enables him to do this. Thus, the proof of Chetayev's Theorem actually implies a stronger assertion than that reflected in its formulation.

2. The assumption that $H(\mathbf{q},\mathbf{p}) \in C_q^2 (D \subset R^{2n})$, which guarantees the uniqueness of the solution, is not essential. Theorem 1 remains valid if $H(\mathbf{q},\mathbf{p}) \in C_q^1$, as follows directly from the scheme of the proof. The condition $H(\mathbf{q},\mathbf{p}) \in C_q^2$ may rather be evaluated in this connection as a consequence of Newton's determinacy principle, as usual in classical mechanics.

3. Comparing the formulations of Theorem 1 and the Corollary, one cannot help noticing that Theorem 1 does not require the assumption $\mathbf{f}(0) = 0$, i.e., the vector field $\mathbf{f}(\mathbf{q})$ may contain a constant component. The possibility of a field structure $\mathbf{f}(\mathbf{q})$ with $\mathbf{f}(0) \neq 0$ is illustrated in the following example.

Example. Consider a system with two degrees of freedom, whose Hamiltonian is defined by

$$H(q_1, q_2, p_1, p_2) = {}^1/_2 (p_1^2 + p_2^2) + \Pi(q_1, q_2)$$
$$\Pi(q_1, q_2) = q_1^4 + q_2^7 + q_1^2 q_2^6$$

Since the potential energy $\Pi(q_1, q_2)$ of the system is not a minimum at $\mathbf{q} = 0$ - an equilibrium position of the system (provided that $\mathbf{p} = 0$), it follows that

$$\omega_\varepsilon = \{(q_1, q_2) \in s_\varepsilon : q_1^4 + q_2^7 + q_1^2 q_2^6 < 0\} \neq \varnothing$$

By the definition of ω_ε, $q_2 < 0$, $\forall (q_1, q_2) \in \omega_\varepsilon$.

Defining the field $\mathbf{f}$ as the vector $\mathbf{f} = (q_1, -1 + q_2)$, we obtain the equality

$$\mathbf{f}^T \partial\Pi/\partial\mathbf{q} = -7q_2^6 - 6q_1^2 q_2^5 + q_1(4q_1^3 + 2q_1 q_2^6) + q_2(7q_2^6 + 6q_1^2 q_2^5)$$

On this basis, noting that $q_1^4 < q_2^7$, $\forall (q_1, q_2) \in \omega_\varepsilon$, so that $|q_1| < |q_2|^{7/4}$ in ω_ε, we get

$$\mathbf{f}^T \partial\Pi/\partial\mathbf{q} = -7q_2^6 + O(|q_2|^7)$$

Thus, for sufficiently small $\varepsilon > 0$,

$$\mathbf{f}^T \partial\Pi/\partial\mathbf{q} < -q_2^6, \ \forall (q_1, q_2) \in \omega_\varepsilon$$

Noting that condition 2 of theorem 1 is certainly satisfied here, we conclude that the equilibrium position $\mathbf{q} = \mathbf{p} = 0$ is unstable.

Applying Poincaré's Recurrence Theorem (/7/, p.447), let us estimate the Lebesgue measure of the trajectories of system (1) that preserve the domain $\Omega_\eta = \Omega_\varepsilon \cap s_\eta^*$ in which, as seen in the proof of Theorem 1, inequality (5) holds.

Theorem 2. Under the assumptions of Theorem 1, if the domain $\Omega_\eta = \{(\mathbf{q},\mathbf{p}) \in s_\eta^*: \ H < 0\}$ contains an invariant set M, then

$$\operatorname{mes}(M) = 0$$

Proof. Suppose that $\operatorname{mes}(M) = \mu > 0$. Then, since the phase volume of system (1) is an invariant and M is a bounded set, it follows from Poincaré's Theorem that almost all trajectories $\gamma \subset M$ possess the recurrence property. Thus there exists a sequence $\{t_m\}$ $(m = 0, 1, 2, \ldots)$ such that

$$\lim_{m\to\infty} t_m = \infty, \quad \lim_{m\to\infty} \|\mathbf{q}(t_m, \mathbf{q}_0, \mathbf{p}_0) \oplus \mathbf{p}(t_m, \mathbf{q}_0, \mathbf{p}_0)\| = \|\mathbf{q}_0 \oplus \mathbf{p}_0\|, \quad \forall (\mathbf{q}_0, \mathbf{p}_0) \in M \setminus \varkappa, \quad t_0 = 0 \tag{9}$$

where

$$\operatorname{mes}(M \setminus \varkappa) = \operatorname{mes}(M)$$

Taking into consideration that the Lebesgue measure of the set

$$\Omega_0 = \{(\mathbf{q},\mathbf{p}) \in s_\eta^*: \ H < 0, \ \mathbf{p} = 0\}$$

is zero, we also have

$$\operatorname{mes}[M^* = (M \setminus \varkappa) \setminus \Omega_0] = \operatorname{mes}(M) \tag{10}$$

Integrating inequality (5) along trajectories $\gamma \subset (M \setminus \varkappa)$, we obtain

$$V(\mathbf{q}(t_m, \mathbf{q}_0, \mathbf{p}_0), \mathbf{p}(t_m, \mathbf{q}_0, \mathbf{p}_0)) - V(\mathbf{q}_0, \mathbf{p}_0) \geqslant c_1 \int_0^{t_m} \|\mathbf{p}(\tau)\|^2 d\tau \tag{11}$$

Putting $(\mathbf{q}_0, \mathbf{p}_0) \in M^*$ in (11), we let m go to infinity. Then, by (9), we get

$$\lim_{m\to\infty} V(\mathbf{q}(t_m, \mathbf{q}_0, \mathbf{p}_0), \mathbf{p}(t_m, \mathbf{q}_0, \mathbf{p}_0)) - V(\mathbf{q}_0, \mathbf{p}_0) = 0$$

$$\lim_{m\to\infty} \int_0^{t_m} \|\mathbf{p}(\tau)\|^2 d\tau > a > 0, \quad a = \text{const}$$

Comparing these relationships we conclude that for sufficiently large m inequality (11) is contradictory whenever $(\mathbf{q}_0, \mathbf{p}_0) \in M^*$. Thus, in view of (10), our assumption that the measure of M is positive is false. This completes the proof of Theorem 2.

Corollary. Almost all trajectories of system (1) that pass through the domain Ω_η intersect the sphere $\overline{s_\eta^*} \setminus s_\eta^*$.

Summarizing, we observe that the restrictions on the structure of the set of critical points of $\Pi(\mathbf{q})$ stipulated in /6/ were motivated by the need to prove the existence of a motion of system (1) for which the derivative $V^\cdot$ satisfies an inequality $V^\cdot \geqslant k > 0$, $k = \text{const}$. As shown above, the inequality $\mathbf{f}^T\, \partial\Pi/\partial\mathbf{q} \leqslant 0$ (together with the other conditions of Theorem 1 and its Corollary) is sufficient to guarantee the existence of such a motion, provided that initial data are chosen subject to the condition $V(\mathbf{q}_0, \mathbf{p}_0) > 0$. The reader will readily convince himself that this condition imposes no restrictions on the structure of the set of critical points of $\Pi(q)$.

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